ANALYTIC HYPOELLIPTICITY FOR $\Box_b + c$ ON THE HEISENBERG GROUP: AN L^2 APPROACH

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ABSTRACT. In an interesting note, E.M. Stein observed some 20 years ago that while the Kohn Laplacian $\Box_b$ on functions is neither locally solvable nor (analytic) hypoelliptic, the addition of a non-zero complex constant reversed these conclusions at least on the Heisenberg group, and Kwon reproved and generalized this result using the method of concatenations. Recently Hanges and Cordaro have studied this situation on the Heisenberg group in detail. Here we give a purely L^2 proof of Stein's result using the author's now classical construction of $(T^p)_\phi = \phi T^p + \ldots$, where T is the 'missing direction' on the Heisenberg group.

1. Introduction

As had been known for many years, the Kohn Laplacian behaves very differently on functions and on forms. This is clearly stated when the underlying manifold is the (complex) Heisenberg group, which we now define.

On $R^{2n-1} \sim C^{n-1} \times R$ we consider the complex vector fields

$$L_j = \frac{\partial}{\partial z_j} - \frac{i}{2}\overline{z_j}\frac{\partial}{\partial t}$$

which have the commutation relations

$$[L_j, L_k] = 0, \quad [L_j, \overline{L_k}] = \delta_{jk} i \frac{\partial}{\partial t} = -\delta_{jk} T.$$

The Kohn Laplacian is defined (on functions) by the expression

$$\Box_b = \sum_j \overline{L_j}^* \, \overline{L_j}$$

and on forms by

$$\Box_b = \overline{\partial}_b^* \overline{\partial}_b + \overline{\partial}_b \overline{\partial}_b^*$$

where $\overline{\partial}_b$ on is the usual extension to forms of the operator

$$v \to \overline{\partial}_b v = (\overline{L}_1 v, \ldots \overline{L}_{n-1} v)$$

to form a complex: $\overline{\partial}^2 = 0$, and * denotes L^2 adjoint. (These formulas are valid quite generally on any CR manifold M where the $\{L_j\}$ form a basis for $T^{1,0}$ in the splitting $CTM = T^{1,0} + T^{0,1} + F$, with $T^{0,1} = \overline{T}^{1,0}$ and $\dim_R(F = \overline{F}) = 1$.)

Thus on the Heisenberg group (i.e., with the vector fields L_j as above), on functions

$$\Box_b = -\sum_j L_j \overline{L}_j.$$

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Now while for a strictly pseudo-convex CR manifold in general, and on the Heisenberg group in particular, $\Box_b$ is hypoelliptic and even analytic hypoelliptic on forms of degree at least one, this is not true on functions and this operator is not even locally solvable.

However in a striking note in 1982 ([3]), E.M. Stein showed that the addition of an arbitrary non-zero complex constant to $\Box_b$ reverses this conclusion and $\Box_b + c$ becomes locally solvable and (analytic-)hypoelliptic. His proof uses explicit kernels.

Soon after, Kwon ([2]) published a paper which, using the method of concatenations of F. Treves, extended the result to a non-zero real analytic *function* $c(z, \overline{z}, t)$ instead of a constant. Recently Cordaro and Hanges have been studying this situation in detail ([1]).

Here we give a totally elementary proof of the result on the Heisenberg group.

2. The *a priori* estimate

The starting point of any local solvability or hypoellipticity proof is a suitable *a priori* estimate. For the operator

$$\Box_b + c = -\sum_j L_j \overline{L}_j + c,$$

we have

$$((\Box_b + c)v, v)_{L^2} = \sum_j \|\overline{L_j} v\|^2_{L^2} + c\|v\|^2_{L^2},$$

so that if 1) $\Re c > 0$ we may assert

$$|\Re((\Box_b + c)v, v)_{L^2}| \geq \sum_j \|\overline{L_j} v\|^2_{L^2} + |\Re c| \|v\|^2_{L^2},$$

while we always have

$$|\Im((\Box_b + c)v, v)_{L^2}| = |\Im c| \|w\|^2_{L^2}$$

and

$$|\Re((\Box_b + c)v, v)_{L^2}| \geq \sum_j \|\overline{L_j} v\|^2_{L^2} - |\Re c| \|v\|^2_{L^2},$$

and so if $|\Re c| < \tilde{d} |\Im c|$ (and $\Re c < 0$):

$$|\Im((\Box_b + c)v, v)_{L^2}| + \tilde{d} |\Re((\Box_b + c)v, v)_{L^2}| \geq \tilde{d} \sum_j \|\overline{L_j} v\|^2_{L^2} + (1 - \tilde{d}) \|v\|^2_{L^2}.$$

Thus for all c except $c < 0$, and suitable $d = d(c)$,

$$|((\Box_b + c)v, v)_{L^2}| \geq C'_c (\sum_j \|\overline{L_j} v\|^2_{L^2} + \|v\|^2_{L^2})$$

uniformly in $v \in C_0^\infty$.

Finally, since the spectrum of $\Box_b$ is discrete, the case $c < 0$ leads to a finite dimensional kernel of $\Box_b + c$ which may be handled by a norm of negative order. Thus for any complex $c \neq 0$ there exists a constant C_c such that for all smooth v of compact support in a neighborhood of the origin,

$$\sum_j \|\overline{L_j} v\|^2_{L^2} + \|v\|^2_{L^2} \leq C_c \{ |((\Box_b + c)v, v)_{L^2}| + \|v\|^2_{-1} \}.$$

Remark 1. The estimate is also valid, with essentially the same derivation, if the non-zero constant c added to $\Box_b$ is replaced by any elliptic pseudodifferential operator of arbitrary order $s < 2$, with the L^2 norm on the left replaced by the norm in H^s and the second norm on the right by any relatively compact norm, for instance, the norm $\| \cdot \|_{H^{s-1}}$.

3. The definition and properties of $(T^p)_\phi$

There are several ways to localize derivatives - when the derivatives are measured by vector fields over which we have maximal control in the estimates, or their conjugates, simple multiplication by a cut-off function of Hörmander-Ehrenpreis type will suffice:

Definition 1. There exists a constant $C = C(n)$ depending only on the dimension n with the following properties: given two bounded open set Ω_j with $\overline{\Omega_1} \subset \Omega_2$, and separation $d = dist(\Omega_1, \Omega_2^c)$, and any natural number N, there exists $\Psi_N \in C_0^\infty(\Omega_2)$, identically equal to one on $\overline{\Omega_1}$, and such that for $|\alpha| \leq 3N$,

$$|D^\alpha \Psi_N| \leq C(\frac{C}{d})^{|\alpha|} N^{|\alpha|}, \quad |\alpha| \leq 3N.$$

The construction of such functions is just a convolution of N identical 'bump' functions with the characteristic function of a set 'midway' between the two sets Ω_j. The factor 3 is of course arbitrary - any other fixed multiple (independent of N) would work as well, with a different choice of $C(n)$.

The rough idea, elaborated below, is that when instead of v, the function $\Psi_N \overline{L}^\alpha u$ is subjected to the *a priori* estimate, brackets of $\Box_b$ with $\Psi_N \overline{L}^\alpha$ will give two kinds of terms - but at least those where the derivatives in the differential operator land on Ψ_N will introduce a factor of CN/d but decrease α by one (counting the derivatives in $\Box_b$ as $\overline{L}$'s for the moment), and after N iterations the factor will be $(CN/d)^N$ which is bounded by Stirling's formula by $(C')^N N!$ leading to analyticity.)

However when the derivative whose high powers (here T^p) are being localized is not one optimally handled by the *a priori* estimate, this procedure is not sufficient. For incurring a derivative on Ψ_N, and hence a factor of CN/d, is not offset by 'gaining' one of the $\overline{L}$'s - since now we do not have a large supply of $\overline{L}$'s to use up, and to convert a T to *two* $\overline{L}$'s (actually one L and one $\overline{L}$) is too costly - it would introduce *two* factors of CN/d to decrease p by one, leading to $C^N N!^2$, the second Gevrey class ($|D^r u| \leq C^{r+1} r!^2$) and not the analytic, or first, Gevrey class.

To localize T^p, since

$$[L_j, \Psi T^p] = (L_j \Psi) T^p,$$

on the Heisenberg group, we use the excellent commutation relations, in particular that $[L_j, \overline{L_k}] = \delta_{jk} T$, all other brackets being zero, to construct

$$T_\Psi = \Psi T + \sum (\overline{L_j} \Psi) L_j - \sum (L_j \Psi) \overline{L_j}$$

which evidently satisfies

$$[L_k, T_\Psi] = \sum_j (L_k \overline{L_j} \Psi) L_j - \sum_j (L_k L_j \Psi) \overline{L_j}$$

and

$$[\overline{L_k}, T_\Psi] = \sum_j (\overline{L_k L_j} \Psi) L_j - \sum_j (\overline{L_k} L_j \Psi) \overline{L_j}.$$

At first sight this may not appear to be much of an improvement over brackets with ΨT alone, but if we observe that here the number of derivatives on the right on Ψ should not be taken as *double* the loss of T derivatives (which would again lead to the second Gevrey class) but rather as *one more* than the loss of T derivatives, we are encouraged to try to generalize this construction for higher powers of T.

Order matters here, and after much trial and error it was found that the following is one eminently satisfactory localization of T^p :

$$(T^p)_\Psi = \sum_{|\alpha+\beta|\le p} \frac{(-1)^{|\alpha|}(L^\alpha \overline{L}^\beta \Psi)}{\alpha!\beta!} L^\beta \overline{L}^\alpha T^{p-|\alpha+\beta|}.$$

For this localization we have

$$[L, (T^p)_\Psi] \equiv 0$$

and

$$[\overline{L_k}, (T^p)_\Psi] \equiv (T^{p-1})_{T\Psi} \circ \overline{L_k}$$

modulo C^p terms of the form

$$\frac{(ML^\alpha \overline{L}^\beta \Psi)}{\alpha!\beta!} L^\beta \overline{L}^\alpha$$

with $|\alpha + \beta| = p$ and $M = L$ or $M = \overline{L}$; we will write this error as

$$C^{P+1}\Psi^{(p+1)}M^p/p!$$

with each instance of $M = L$ or $M = \overline{L}$.

If we apply these elegant relations to our solution via the *a priori* estimate for $v = (T^p)_\Psi u$ and $(\Box_b + c)u = f \in C^\omega$, we have

$$\begin{aligned}
&\sum_j \|\overline{L_j}(T^p)_\Psi u\|_{L^2}^2 + \|(T^p)_\Psi u\|_{L^2}^2 \le \\
&\le C\{|((\Box_b + c)(T^p)_\Psi u, (T^p)_\Psi u)_{L^2}| + \|(T^p)_\Psi u\|_{-1}^2\} \\
&\lesssim |((T^p)_\Psi f, (T^p)_\Psi u)_{L^2}| + \|(T^p)_\Psi u\|_{-1}^2 + \\
&+ |(\sum_k [L_k \overline{L_k}, (T^p)_\Psi]u, (T^p)_\Psi u)_{L^2}|,
\end{aligned}$$

where the notation $A \lesssim B$ denotes $A \le CB$ with a constant C depending only on the dimension n.

Thus, expanding the bracket,

$$\begin{aligned}
[L_k \overline{L_k}, (T^p)_\Psi]u &= L_k[\overline{L_k}, (T^p)_\Psi]u + [L_k, (T^p)_\Psi]\overline{L_k}u = L_k \circ (T^{p-1})_{T\Psi} \circ \overline{L_k}u \\
&= (T^{p-1})_{T\Psi} \circ L_k \overline{L_k}u
\end{aligned}$$

modulo C^p terms of the form $\Psi^{(p+1)}M^{p+1}/p!$ and in fact, due to the simple form of $\Box_b$, upon summing we recover $\Box_b$, though this will not help us.

Suffice it to say that from this expansion, upon iteration and with the weighted Schwarz inequality we may write:

$$\begin{aligned}
&\sum_j \|\overline{L_j}(T^p)_\Psi u\|_{L^2}^2 + \|(T^p)_\Psi u\|_{L^2}^2 \lesssim \sum_{0\le q\le p} C^q \|(T^{p-q})_{T^q\Psi} f\|_{L^2}^2 + \\
&+ \sum_{1\le q\le p} C^q \|\overline{L_j}(T^{p-q})_{T^q\Psi} u\|_{L^2}^2 + \sum_{1\le q\le p} C^q \|(T^{p-q})_{T^q\Psi} u\|_{L^2}^2 + \\
&+ \sum_{0\le q\le p} C^q \|(T^{p-q})_{T^q\Psi} u\|_{-1}^2 + \sum_{q\ge 0} \Big(C^q \|\Psi^{(p+1)} M^{p-q+1} u\|/(p-q)!\Big)^2
\end{aligned}$$

Now if the support of Ψ is small enough, the -1 norm is less than a small multiple of the L^2 norm, so the next to last term on the right will be absorbed by the second term on the left (and the third term on the right) - the trickier term is the last on the right.

For this term we will introduce a *new* cut-off function, $\tilde{\Psi}$, equal to one on the support of Ψ, and with essentially the same growth of its derivatives (they can be made the same by

taking d half as large originally and nesting three open sets instead of two). Then bring the high derivatives of Ψ out of the norm bounded by $(C/d)^{p+1}N^{p+1} \sim (\tilde{C}/d)^{p+1}p!$ when N is comparable to p. This iteration shows that to bound high T derivatives applied to u locally by the corresponding factorial it will suffice to bound derivatives of the same order but in the 'directions' L and $\overline{L}$ by the same factorials.

4. Mixed powers of L_k and $\overline{L_j}$

In this section we consider mixed L and $\overline{L}$ derivatives of u. We claim that judicious integration by parts will reduce us to consideration of pure powers of $\overline{L}$ modulo *half as many* powers of T. For this, we need to treat expressions such as

$$(L^\alpha \overline{L}^\beta w, L^\alpha \overline{L}^\beta w)_{L^2} - (\overline{L}^\alpha \overline{L}^\beta w, \overline{L}^\alpha \overline{L}^\beta w)_{L^2} = \pm([\overline{L}^\alpha, L^\alpha]\overline{L}^\beta w, \overline{L}^\beta w)_{L^2}$$

for $w \in C_0^\infty$.

Now an elementary calculation expresses $[\overline{L}^\alpha, L^\alpha]$ as a sum

$$[\overline{L}^\alpha, L^\alpha] = \sum_{0 \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'}^2 \alpha'! T^{|\alpha'|} L^{\alpha - \alpha'} \overline{L}^{\alpha - \alpha'},$$

and then we merely integrate $L^{\alpha-\alpha'}$ and approximately *half* of the T's by parts in the inner product.

Howerver, to sum things up so far, we note (taking $f = 0$ for simplicity) that iterating the *a priori* estimate above and bringing $\Psi^{(p+1)}$ out of the norm, we arrive, for p comparable to N, at

$$\frac{\sum_j \|\overline{L_j}(T^p)_\Psi u\|_{L^2} + \|(T^p)_\Psi u\|_{L^2}}{p!} \leq$$

$$\leq (\tilde{C}/d)^p p! \sup_{p \geq q+2r \geq 0} C^q r! \frac{\|\tilde{\Psi}\overline{L}^{p-q-2r+1}T^r u\|}{(p-q)!}$$

or, more suggestively, in the region where $\Psi = \Psi_1 \equiv 1$, and writing d_1 for that d,

$$\frac{\sum_j \|\overline{L_j}T^p u\|_{L^2(\Psi_1 \equiv 1)} + \|T^p u\|_{L^2(\Psi_1 \equiv 1)}}{p!} \leq$$

$$\leq (C/d_1)^p \sup_{p \geq q+2r \geq 0} C^q \frac{\|\overline{L}\Psi_2 \overline{L}^{p-q-2r}T^r u\|}{(p-q-r)!}$$

with $\Psi_2 \equiv 1$ on the support of Ψ_1. Note that the order of T derivatives here can not exceed half of the order on the left hand side, and the same will be true even after brackets when brackets of L's and $\overline{L}$'s produce additional T's.

5. Brackets with $\overline{L}^\alpha$ (and a few T's and perhaps one L)

The analysis of powers of $\overline{L}$ is relatively straightforward, since we have a supply of 'good' vector fields to use over and over in the *a priori* estimate. All that happens is that in the bracket with $\Box_b(+c)$, when L meets $\overline{L}^{p-q-2r}$ there may be as many as $p-q-2r$ copies of T and the total number of $\overline{L}$'s will be decreased by two:

$$([L_k\overline{L_k}, \Psi_2\overline{L}^s]T^m u, \Psi_2\overline{L}^s T^m u)_{L^2} \sim (\Psi_2\overline{L_k L}^s T^m u, \Psi_2'\overline{L}^s T^m u)_{L^2} +$$

$$+(\Psi_2'\overline{L}^s T^m u, \overline{L_k}\Psi_2\overline{L}^s T^m u)_{L^2} + s(\Psi_2\overline{L}^s T^{m+1}u, \Psi_2\overline{L}^s T^m u)_{L^2}$$

(and some terms with two derivatives on Ψ, and a drop of two $\overline{L}$'s - in fact we may here safely ignore derivatives on Ψ, and for instance we have interchanged the two localizing functions in the first term on the right).

To handle the last term we do not try to imagine half a power of T belonging to each side of the inner product, though we could do so using pseudodifferential operators, but rather do the paradoxical manoever:

$$s(\Psi_2\overline{L}^s T^{m+1}u, \Psi_2\overline{L}^s T^m u)_{L^2} \sim s(\Psi_2\overline{L}^{s-1}T^{m+1}u, L\Psi_2\overline{L}^s T^m u)_{L^2}$$
$$\sim s\|\overline{L}\Psi_2\overline{L}^{s-2}T^{m+1}u\|_{L^2}\left(\|\overline{L}\Psi_2 L\overline{L}^{s-1}T^m u\|_{L^2} + \|\Psi_2\overline{L}^{s-1}T^{m+1})u\|_{L^2}\right) \sim$$
$$\sim C_\varepsilon(s\|\overline{L}\Psi_2\overline{L}^{s-2}T^{m+1}u\|_{L^2})^2 + \varepsilon\|\overline{L}\Psi_2 L\overline{L}^{s-1}T^m u\|^2_{L^2}$$

for any $\varepsilon > 0$ (plus terms with Ψ_2 differentiated and corresponding gains.)

While the last term on the right has one L derivative, normally not well estimated, we have seen that we are able to iterate the *a priori* estimate as long as one $\overline{L}$ remains.

To see that this is the case, we begin the iteration of the *a priori* estimate with the function $v = \Psi L\overline{L}^{s-1}T^m u$ instead of $\Psi\overline{L}^s T^m u$. The corresponding bracket to consider is (modulo Ψ'):

$$([L\overline{L}, \Psi_2 L\overline{L}^{s-1}]T^m u, \Psi_2 L\overline{L}^{s-1}T^m u)_{L^2} \sim$$
$$\sim s(\Psi L\overline{L}^{s-1}T^{m+1}u, \Psi L\overline{L}^{s-1}T^m u)_{L^2}$$
$$\sim s(\Psi\overline{L}^{s-1}T^{m+1}u, \overline{L}L\Psi\overline{L}^{s-1}T^m u)_{L^2}$$

which, after a weighted Schwarz inequality, reduces to the previous case (iterated one step) and a small multiple of the left hand side.

To conclude the proof we observe that so far we have passed from estimating p derivatives in Ω_0 to estimating $p/2$ derivatives in Ω_1 :

$$\frac{\|D^p u\|_{L^2(\Omega_0)}}{p!} \le (C_0/d_0)^p \sup_{\tilde{p}\le p/2} \frac{\|D^{\tilde{p}}u\|_{L^2(\Omega_1)}}{\tilde{p}!}$$

with d_0 equal to the separation between Ω_0 and the complement of Ω_1.

Lastly we need to nest $\log_2 p$ open sets between Ω_0 and the set $\tilde{\Omega}$ where f is assumed to be analytic with separations $d_0, d_1, \dots d_{\log_2(p)}$ in such a way that the sum is bounded independently of p and the product is less than a universal constant raised to the power p, e.g., such that

$$\sum_0^{\log_2(p)} d_j \le 1, \qquad \prod_0^{\log_2(p)} d_j^{-p/2^j} \le C^p.$$

But $d_j = 1/2^{j+1}$ will satisfy these conditions.

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